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Combinatorical aspects of the Schwinger–Dyson equation ★

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Abstract

In this work we analyse combinatorical aspects of the Schwinger–Dyson equation. This leads to generalizations of Wick's theorems on integrals with Gaussian weight to a larger class of weights which we call sub-Gaussian. Examples of sub-Gaussian contractions are that of Kac–Moody or Virasoro type, although the concept of a sub-Gaussian weight does not refer a priori to two-dimensional field theory. The generalization was chosen in such a way that the contraction rules become a combinatorical way of solving the Schwinger–Dyson equation. In a still more general setting we prove a relation between solutions of the Schwinger–Dyson equation and a map N, which in the Gaussian case reduces to normal ordering. Furthermore, we give a number of results concerning contractions of composite insertions.

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1. Introduction

1.1. *The Schwinger–Dyson equation*

In this paper we will be concerned with the Schwinger–Dyson equation: $\forall_{i,f}\ I(\partial_i(S)f) = I(\partial_i f)$, which for fixed $S : \mathbb{R}^D \to \mathbb{R}$ is the equation satisfied by

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the functional

$$f \mapsto I(f) := \int_{\mathbb{R}^D} f\mathrm{e}^{-S}\,\mathrm{d}x^1 \ldots \mathrm{d}x^D,$$

where f and S are restricted such that it is well defined, and such that upon partial integration boundary terms are zero: [2]

$$0 = \int \frac{\partial}{\partial x^i}(\mathrm{e}^{-S} f)\,\mathrm{d}x^{1\ldots D} = \int \mathrm{e}^{-S}(-\partial_i(S)f + \partial_i f)\,\mathrm{d}x^{1\ldots D} = I(\partial_i f - \partial_i S).$$

The interest of this equation is that it can be generalized to infinite dimensions by replacing ∂_i by functional derivation. Thus, positive solutions of this equation can be used as guidelines to construct infinite-dimensional measures. We set $\langle f\rangle := I(f)/I(1)$.

1.2. Aim and overview

The above is a motivation to study the Schwinger–Dyson equation in general; in this article we will restrict our attention to the following points:

(i) The first aim of this work is to look for conditions for the action S under which the Schwinger–Dyson equation has a unique solution up to normalization.
For example, Gaussian weights have a unique solution, but as we will see there is more. In order to be able to solve the Schwinger–Dyson equation by combinatorics, we will impose a restriction on the possible actions that we consider, namely that they satisfy a differential equation of the form

$$\partial_i\partial_j S = g_{ij} + \Gamma_{ij}^k(\partial_k S) + \Delta_{ij}^{kl}(\partial_k S)(\partial_l S) + \cdots,$$

stopping after a finite number of terms. Obviously quadratic actions are in this class. The interest of this equation comes from considering the Schwinger–Dyson equation in new variables $\partial_i S$; It then reads:

$$\langle \partial_{i_1}(S)\ldots\partial_{i_n}(S)\rangle = \sum_{k=2}^{n}\langle \partial_{i_2}(S)..\partial_{i_1}\partial_{i_k}(S)..\partial_{i_n}(S)\rangle$$

so that if $\partial^2 S$ is expressible in ∂S, then at least the equation closes.

(ii) Next we aim to extend the notion of normal ordering to non-Gaussian weights in such a way that it is *naturally* associated to such weights, since when using functional integration for geometric purposes, it is essential to only use natural constructions.
Our definition will be by induction, using the new variables $S_i := \partial_i S$, as follows: $N(1) := 1$, and

$$N(S_{i_0}..S_{i_n}) := S_{i_0}N(S_{i_1}..S_{i_n}) - \frac{\partial}{\partial x^{i_0}}N(S_{i_1}..S_{i_n}).$$

[2] We will stick to the practice, in the context of functional integration, of calling this formula the Schwinger–Dyson equation, although the above formula was already given for functional integrals by Feynman in [5, formula 45]. Another often seen formulation is $\{J_i - (\partial_i S)(\partial/\partial J)\}Z(J) = 0$, by setting $Z(J) := I(\mathrm{e}^{J_i x^i})$.

In the Gaussian case this reduces to usual[3] normal ordering. The Schwinger–Dyson equations leads to the above non-Gaussian extension in the sense that solutions of the Schwinger–Dyson equation can be expressed as a linear function of N^{-1}. We will generalize this remark to the case where the vector fields ∂_i are replaced by a not necessarily Abelian Lie algebra of vector fields on a manifold. It is in this non-Abelian case that non-Gaussian examples are known.

(iii) Ordering the possible differential equations for S by the highest occurring power of ∂S, the first equation is $\partial_i \partial_j S = g_{ij}$, i.e. leading to Gaussian weights; the next one, at most linear in ∂S will be called sub-Gaussian. We will prove a number of generalizations of Wick's theorems [16] to that case; the interest of such theorems being that they provide an algebraic setting for solving the Schwinger–Dyson equation, a setting in which the fields $\phi(x)$ are just symbols instead of operator-valued distributions acting on Hilbert spaces. Unlike other suggestions for non-Gaussian Wick rules, our derivation is not restricted to a specific underlying dimension. These Wick rules will be illustrated through examples from two-dimensional field theory.

(iv) Finally we will look for theorems concerning "composite insertions", by which we will mean factors in expressions between brackets $\langle\cdot\rangle$ which are not first derivatives of S: The fact that in $\langle\partial_i(S)s_1..s_n\rangle$ we may eliminate $\partial_i(S)$ in favour of the sum of terms with $\partial_i s_j$ relies on the special form of $\partial_i S$; in general it will not be possible to find a derivation D such that $\langle X(S)Y(S)f\rangle = \langle Df\rangle$. However it may happen sometimes, if we choose the vector fields X and Y in the right way. We will be proving some theorems concerning the conditions under which this happens.

The organization of the article is as follows: In Section 2 we will give precise definitions of what we mean by the non-Abelian sub-Gaussian case, and prove a number of theorems concerning them. Instead of speaking of actions S we will phrase everything in terms of contractions $[\cdot \triangleright \cdot]$, which is a formulation better suited to study the Schwinger–Dyson equation from the combinatorical point of view. In that section we will also review a relatively well-known algebra as an example of a non-Abelian contraction: The Kac–Moody algebra. In Section 3 we will be concerned with composite insertions, and we will see how the solution of the Schwinger–Dyson equation for one system can be helpful to solve the Schwinger–Dyson for another one. We will treat a classical example where this is the case in detail, the Sugawara construction. The example is not new: What we want to emphasize is their relation with the Schwinger-Dyson equation, and the use of sub-Gaussian calculus. In Section 4, we will be concerned with the proof in the case of possibly non-commuting vector fields that the solutions $\langle\cdot\rangle$ of the Schwinger–Dyson equation are given by ZN^{-1}, where Z is known explicitly.

[3] Gaussian normal ordering was introduced in [7]. A number of definitions can be found for other cases besides the Gaussian case, see for example: [13; 10, formula 7b; 3, formula 4; 17, formula 6]. However none of these definitions is directly in terms of the action S.

2. Contraction algebras

In this section we will introduce an algebraic structure which we call a contraction algebra. It is an abstraction of an action S with the property that $\partial^2 S$ can be expressed as a polynomial in ∂S, i.e. satisfying a differential equation of the form $\partial_i \partial_j S = g_{ij} + \Gamma_{ij}^k (\partial_k S) + \Delta_{ij}^{kl} (\partial_k S)(\partial_l S) + \cdots$. Indeed, to such an identity we can associate the following binary operation on the vector fields ∂_i:

$$[\partial_i \triangleright \partial_j] := g_{ij} + \Gamma_{ij}^k \partial_k + \Delta_{ij}^{kl} \partial_k \partial_l + \cdots$$

with values in symmetric polynomials in the symbols ∂_i. This operation satisfies properties that we will use as axioms for the definition of contraction algebra.

The advantage of using the notion of a contraction $[\cdot \triangleright \cdot]$ over that of an action S is that $[\cdot \triangleright \cdot]$ contains the only necessary data for the *combinatorical* solutions we are looking for; it may happen that a contraction is known but that the corresponding action is not, since easy differential equations can have difficult solutions. Also, as we will see, the statement of generalized Wick theorems is easier in this language.

Given this contraction, we can now altogether omit S from the Schwinger–Dyson equation $\langle \partial_{i_1}(S) \ldots \partial_{i_n}(S) \rangle = \sum_{k=2}^n \langle \partial_{i_2}(S)..\partial_{i_1}\partial_{i_k}(S)..\partial_{i_n}(S) \rangle$ and just write

$$\langle \partial_{i_1} \ldots \partial_{i_n} \rangle = \sum_{k=2}^{n} \langle \partial_{i_2}..[\partial_{i_1} \triangleright \partial_{i_k}]..\partial_{i_n} \rangle,$$

which is the equation we will analyse.

Next we will need a generalization of the contraction to the case where the vector fields ∂_i are replaced by a basis $\{T_a\}$ of a not necessarily Abelian Lie algebra of vector fields. In that case, the contraction roughly means that $T_a T_b S$ is expressed in terms of $T_a S$, but not exactly: When using general vector fields X on some manifold M, we can no more use the fact that the measure $\mathrm{d}x^1..\mathrm{d}x^n$ on $\mathbb{R}^n$ is invariant under the vector fields ∂_i that we used before. Therefore there is no point in that case in splitting off the weight e^{-S} from the volume form $\mu = \mathrm{e}^{-S}\mathrm{d}x^1..\mathrm{d}x^n$, so that we will only talk about the volume form μ from now on. By taking Lie derivatives of the integrand, the Schwinger–Dyson equation then reads: $I(X(f) + f\nabla(X)) = 0$, where the divergence is defined by $L_X \mu = \nabla(X)\mu$. When specializing to $\mu = \mathrm{e}^{-S}\mathrm{d}x^1..\mathrm{d}x^n$ and $X = \partial_i$, we see that $\nabla(X) = -X(S)$, so instead of assuming that $T_a T_b(S)$ can be expressed as a polynomial in $T_c(S)$, we will rather assume that $T_a \nabla(T_b)$ is uniquely expressible in $\nabla(T_c)$'s.

Given such a divergence ∇ we may define a map $[\cdot \triangleright \cdot] : L \otimes L \to Sym(L)$; $[X \triangleright Y] := -X(\nabla(Y))$, which we extend on the right by derivations to $Sym(L)$. Thus in this notation one may think of $X \in Sym(L)$ as $X(S)$. This map will automatically satisfy indentities which are stated in Definition 1(i) below, by the general property of divergences that $\nabla([X, Y]) = X(\nabla(Y)) - Y(\nabla(X))$. The sub-Gaussian case, i.e. where $\partial^2 S$ was at most linear in ∂S, now corresponds to the contraction being a map $L \otimes L \to K \oplus L \leq Sym(L)(K = \text{scalars})$, since L corresponds to the first derivatives of S.

Definition 1. We define a number of special contraction algebras:

(i) A polynomial contraction algebra is a Lie algebra L with a map $[\cdot \rhd \cdot] : L \otimes L \to Sym(L)$, extended by derivations on the right to $Sym(L)$, which satisfies

$$[X \rhd Y] - [Y \rhd X] = [X, Y] \in L \leq Sym(L),$$
$$[X \rhd [Y \rhd Z]] - [Y \rhd [X \rhd Z]] = [[X, Y] \rhd Z].$$

(ii) It is called Gaussian if $[\cdot \rhd \cdot] : L \otimes L \to K = Sym^0(L)$ (=scalars).

(iii) A sub-Gaussian contraction algebra is one in which $[\cdot \rhd \cdot] : L \otimes L \to K \oplus L = Sym^{[0,1]}(L)$. In that case, we extend $[\cdot \rhd \cdot]$ by $[1 \rhd 1] := [1 \rhd X] := 0$, which makes $K \oplus L$ into a pre-Lie algebra. [4]

(iv) Normal ordering is the map $N : Sym(L) \to Sym(L)$, defined by $N(1) := 1$, and by

$$N(X_1 \dots X_n) := \frac{1}{n} \sum_{i=1}^{n} X_i N(X_1 .. \hat{X}_i .. X_n) - [X_i \rhd N(X_1 .. \hat{X}_i .. X_n)].$$

(v) The contraction is said to be non-degenerate iff its normal ordering is invertible.

(vi) in that case we set $s_1 * s_2 := N^{-1}(N(s_1)N(s_2))$ which is an associative symmetric product: $Sym(L) \otimes Sym(L) \to Sym(L)$.

(vii) For maps $\langle \cdot \rangle : Sym(L) \to K$, we now define the Schwinger–Dyson equation to be the equation $\forall_{X \in L; s \in Sym(L)} \langle Xs \rangle = \langle [X \rhd s] \rangle$.

2.1. Sub-Gaussian contraction algebras

This section contains theorems valid only for sub-Gaussian algebras. A simple example of a sub-Gaussian contraction is the one-dimensional Abelian Lie algebra with basis element e, and contraction $[e \rhd e] := \lambda 1 + \mu e$ for some scalars λ and μ. We will see later that infinite-dimensional examples are well known in different guise.

Theorem 2. *Set* $[X \circ Y] := \frac{1}{2}[X \rhd Y] + \frac{1}{2}[Y \rhd X]$. *In sub-Gaussian algebras the following holds:* [5]

(i) $[[X \rhd Y] \circ Z] + [Y \circ [X \rhd Z]] - [X \rhd [Y \circ Z]] = \frac{1}{2}([[Y \rhd X] \rhd Z] + [[Z \rhd X] \rhd Y])$,

(ii) $N([X \rhd Y]) = [X \rhd Y]$,

(iii) $N([X \rhd Y]Z) = [X \rhd Y]Z - [[X \rhd Y] \circ Z]$,

(iv) $[X \rhd N(YZ)] = N([X \rhd Y]Z + Y[X \rhd Z] + \frac{1}{2}[[Y \rhd X] \rhd Z] + \frac{1}{2}[[Z \rhd X] \rhd Y])$.

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[4] I wish to thank C.D.D. Neumann for pointing out the following to me: a pre-Lie algebra is a vectorspace with a bilinear operation $[\cdot \rhd \cdot]$ satisfying $[a \rhd [b \rhd c]] - [b \rhd [a \rhd c]] = [[a \rhd b] - [b \rhd a] \rhd c]$, see [6, formula 6]. In that case $[a, b] := [a \rhd b] - [b \rhd a]$ is a Lie composition.

[5] For other non-Gaussian wick theorems, valid however only in the context of two-dimensional conformal field theory, see [6, Section 5;1, Appendix A].

Proof.

(i) This is an identity which holds in any pre-Lie algebra:

$$
\begin{aligned}
2\,\mathrm{LHS} &= [[X \triangleright Y] \triangleright Z] + [Z \triangleright [X \triangleright Y]] + [Y \triangleright [X \triangleright Z]] + [[X \triangleright Z] \triangleright Y] \\
&\quad - [X \triangleright [Y \triangleright Z]] - [X \triangleright [Z \triangleright Y]] \\
&= [[Y \triangleright X] \triangleright Z] + [Z \triangleright [X \triangleright Y]] + [Y \triangleright [X \triangleright Z]] + [[Z \triangleright X] \triangleright Y] \\
&\quad - [Y \triangleright [X \triangleright Z]] - [Z \triangleright [X \triangleright Y]] \\
&= [[Y \triangleright X] \triangleright Z] + [[Z \triangleright X] \triangleright Y].
\end{aligned}
$$

(ii) $[X \triangleright Y] \in Sym^{[0,1]}(L)$.

(iii) Idem, together with the symmetrized definition of N.

(iv)
$$
\begin{aligned}
\mathrm{LHS} &= [X \triangleright YZ - [Y \circ Z]] \\
&= [X \triangleright Y]Z + Y[X \triangleright Z] - [X \triangleright [Y \circ Z]] \\
&= N([X \triangleright Y]Z) + N(Y[X \triangleright Z]) \\
&\quad + [[X \triangleright Y] \circ Z] + [Y \circ [X \triangleright Z]] - [X \triangleright [Y \circ Z]] \\
&= \mathrm{RHS}.
\end{aligned}
$$

This completes the proof. □

Note that Theorem 2(iv) is a generalization of the Gaussian fact that $[X \triangleright N(YZ)] = N([X \triangleright YZ])$. In the next section we will make use of this formula to handle expressions like $N(J(z)J(z))$. A more common use of the formula is in Gaussian form: It is then the essential statement for proving that in the Gaussian case $\langle N(X_1X_2)N(Y_1Y_2)\rangle$ say, can be expanded in terms of contractions between X's and Y's only; Indeed,

$$
\begin{aligned}
&\langle N(X_1X_2)N(Y_1Y_2)\rangle \\
&\quad = \langle (X_1N(X_2) - [X_1 \triangleright N(X_2)])N(Y_1Y_2)\rangle \\
&\quad = \langle N(X_2)[X_1 \triangleright N(Y_1Y_2)]\rangle = \langle N(X_2)N([X_1 \triangleright Y_1Y_2])\rangle.
\end{aligned}
$$

Theorem 3 (Sub-Gaussian reconstruction). *Let L be finite-dimensional Lie algebra. Let $\langle\cdot\rangle : Sym(L) \to K$ be the solution of the Schwinger–Dyson equation determined by a sub-Gaussian contraction on L. Then this contraction can be reconstructed from $\langle\cdot\rangle$ if $c_{ij} := \langle T_iT_j\rangle$ is invertible (T_i a basis for L) as follows: Let $c_{ijk} := \langle T_iT_jT_k\rangle$ and $[T_i, T_j] =: f_{ij}^k T_k$. Then the contraction is given by $[T_i \triangleright T_j] = g_{ij} + \Gamma_{ij}^k T_k$, where*

$$
g_{ij} := c_{ij}, \qquad \Gamma_{ij}^k := \tfrac{1}{2} g^{kl}(c_{ijl} + f_{ij}^m g_{ml} - f_{il}^m g_{jm} - f_{jl}^m g_{im}).
$$

Proof. The proof is similar to the uniqueness proof of the Levi-Cività connection, and was inspired by work of Kutasov [12, formula 2]: We know that the contraction is sub-Gaussian, so it is of the form $[T_i \triangleright T_j] = g_{ij} + \Gamma_{ij}^k T_k$, and it remains to prove the above relations. Indeed,

$$c_{ij} = \langle T_i T_j \rangle = \langle [T_i \triangleright T_j] \rangle = \langle g_{ij} + \Gamma^c_{ij} T_c \rangle = g_{ij}.$$

Further, we have

$$c_{ijk} = \langle [T_i \triangleright T_j] T_k \rangle + \langle T_j [T_i \triangleright T_k] \rangle = \Gamma^m_{ij} c_{mk} + c_{jm} \Gamma^m_{ik}.$$

Using the fact that $\Gamma^k_{ij} - \Gamma^k_{ji} = f^k_{ij}$, since $\Gamma^k_{ij} T_k - \Gamma^k_{ji} T_k = [T_i \triangleright T_j] - [T_j \triangleright T_i] = [T_i, T_j]$, we arrive at

$$\begin{aligned} c_{ijk} + c_{jik} - c_{kij} &= \Gamma^m_{ij} c_{mk} + \Gamma^m_{ik} c_{jm} + \Gamma^m_{ji} c_{mk} + \Gamma^m_{jk} c_{im} - \Gamma^m_{ki} c_{mj} - \Gamma^m_{kj} c_{im} \\ &= (\Gamma^m_{ij} + \Gamma^m_{ji}) c_{mk} + f^m_{ik} c_{jm} + f^m_{jk} c_{im} \\ &= (2\Gamma^m_{ij} - f^m_{ij}) c_{mk} + f^m_{ik} c_{jm} + f^m_{jk} c_{im} \end{aligned}$$

So that

$$2\Gamma^l_{ij} g_{lk} = c_{ijk} + c_{jik} - c_{kij} + f^m_{ij} c_{mk} - f^m_{ik} c_{jm} - f^m_{jk} c_{im}.$$

2.2. An example from conformal field theory: The Kac–Moody algebra

Theorem 4. *The following almost everywhere defined contraction satisfies the pre-Lie property and therefore defines a sub-Gaussian contraction algebra: The algebra is associated to a Lie algebra L with invariant symmetric bilinear from h; it is defined by generating symbols $J(X, z)$, linear in $X \in L$, where $z \in \mathbb{C}$, with contraction*:

$$[J(X, z) \triangleright J(Y, \zeta)] := \frac{h(X, Y)}{(z - \zeta)^2} 1 + \frac{J([X, Y], \zeta)}{z - \zeta}.$$

Proof. The pre-Lie property is equivalent to the statement that $[[a \triangleright b] \triangleright c] + [b \triangleright [a \triangleright c]]$ is symmetric in a and b. Note that this way of proving it will automatically give an expression for the three-point function: $\langle abc \rangle = \langle [a \triangleright b] c \rangle + \langle b [a \triangleright c] \rangle = \langle [[a \triangleright b] \triangleright c] + [b \triangleright [a \triangleright c]] \rangle$.

Lemma 5. *Following this remark, we first prove*

$$[[J(X_1, z_1) \triangleright J(X_2, z_2)] \triangleright J(X_3, z_3)] + [J(X_2, z_2) \triangleright [J(X_1, z_1) \triangleright J(X_3, z_3)]] \quad \square$$
$$= \frac{h([X_1, X_2], X_3) + J((z_1 - z_3)[[X_1, X_2], X_3] + (z_1 - z_2)[X_2, [X_1, X_3]], z_3)}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}$$

Proof.

$$\begin{aligned} \text{LHS} &= \left[\frac{J([X_1, X_2], z_2)}{(z_1 - z_2)} \triangleright J(X_3, z_3) \right] + \left[J(X_2, z_2) \triangleright \frac{J([X_1, X_3], z_3)}{(z_1 - z_3)} \right] \\ &= t(h) + t(J), \end{aligned}$$

where $t(h)$ denotes the terms involving h, and $t(J)$ those with J.

$$\begin{aligned} t(h) &= \frac{h([X_1, X_2], X_3)}{(z_1 - z_2)(z_2 - z_3)^2} + \frac{h(X_2, [X_1, X_3])}{(z_1 - z_3)(z_2 - z_3)^2} \\ &= \frac{h([X_1, X_2], X_3)}{(z_2 - z_3)^2} \left(\frac{1}{z_1 - z_2} - \frac{1}{z_1 - z_3} \right) \\ &= \frac{h([X_1, X_2], X_3)}{(z_2 - z_3)^2} \frac{z_2 - z_3}{(z_1 - z_2)(z_1 - z_3)} \\ &= \frac{h([X_1, X_2], X_3)}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}, \end{aligned}$$

and

$$\begin{aligned} t(J) &= \frac{J([[X_1, X_2], X_3], z_3)}{(z_1 - z_2)(z_2 - z_3)} + \frac{J([X_2, [X_1, X_3]], z_3)}{(z_1 - z_3)(z_2 - z_3)} \\ &= \frac{J((z_1 - z_3)[[X_1, X_2], X_3] + (z_1 - z_2)[X_2, [X_1, X_3]], z_3)}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}, \end{aligned}$$

which proves the lemma. □

Proof of Theorem 4 (continued). It remains to prove that the result of Lemma 5 is symmetric under the exchange of 1 and 2. This is clear for the term with h, and for the J term, it suffices to prove that

$$(z_1 - z_3)[[X_1, X_2], X_3] + (z_1 - z_2)[X_2, [X_1, X_3]] + (1 \leftrightarrow 2) = 0.$$

Indeed, using the Jacobi identity:

$$\begin{aligned} \text{LHS} &= \{(z_1 - z_3) - (z_2 - z_3)\}[[X_1, X_2], X_3] \\ &\quad + (z_1 - z_2)\{[X_2, [X_1, X_3]] - [X_1, [X_2, X_3]]\} \\ &= (z_1 - z_2)[[X_1, X_2], X_3] + (z_1 - z_2)[[X_2, X_1], X_3] = 0. \end{aligned}$$

□

Remark 6. In the same way, the reader may check that the Virasoro algebra also defines a sub-Gaussian contraction; for $c \in \mathbb{R}$, this is an algebra generated by symbols $\partial^k T(z)$, where $k \in \mathbb{N}$ and $z \in \mathbb{C}$. The contraction reads

$$[T(z) \rhd T(\zeta)] := \frac{c/2}{(z - \zeta)^4} 1 + \frac{2T(\zeta)}{(z - \zeta)^2} + \frac{\partial T(\zeta)}{(z - \zeta)},$$

together with $[\partial^k T(z) \rhd \partial^l T(w)] := \partial_z^k \partial_w^l [T(z) \rhd T(w)]$.

Definition 7. A module for a pre-Lie algebra is defined to be a module for the induced Lie-algebra. The reader may check that the following operations [2,11] define modules for the Virasoro and Kac–Moody algebras:

$$[T(z) \rhd \phi(\zeta)] := \frac{h.\phi(\zeta)}{(z - \zeta)^2} + \frac{\partial \phi(\zeta)}{(z - \zeta)}; \qquad [J(X, z) \rhd \phi(v, \zeta)] := \frac{\phi(Xv, \zeta)}{z - \zeta}.$$

Here, h is a number, $\zeta \in \mathbb{C}$ and v runs linearly over a representation space of the Lie algebra L. In that case the symbol ϕ is called a primary field for T or J, and the number h is called its conformal weight.

The reader who is familiar with meromorphic operator products in two-dimensional conformal field theory will note that in the above examples, the contraction equals the singular part of the operator product expansion (OPE). That both the contraction and the OPE have the same singular part can be understood if $\langle abc\rangle = \langle [a \triangleright b]c\rangle + b[a \triangleright c]\rangle$ holds, by comparing the singularities between a and b. Thus, a simple guess for contractions is the singular part of the operator product.

This does not mean that the category of contraction algebras is obtained from that of the vertex operator algebras (VOAs) by forgetting the information contained in the regular part of the OPE: First, VOAs are designed especially for two-dimensional field theory, whereas contraction algebras are defined in any dimension. Next, as motivated by functional integral expressions, contraction algebras distinguish between fundamental fields (elements of L) and composite fields (elements of $Sym(L)$), a distinction which is not made in VOAs. Sometimes vertex operators are associated to a contraction algebra, for example starting from the Gaussian contraction $[\phi(x) \triangleright \phi(y)] := \ln|x-y|$ with $x \in \mathbb{R}^2$, one can define the composite fields $V^p_{(I_1,\ldots,I_n)}(x) := N((\partial_{I_1}\phi)(x)..(\partial_{I_n}\phi)(x)e^{ip\phi(x)})$, using the multi-index notation $\partial_J := \partial_{j_1}..\partial_{jk}$, which are closed under Taylor expansion of the product in $S\hat{y}m(L)$: For example, using that for Gaussian contractions $N(e^X)N(e^Y) = N(e^{X+Y+\langle XY\rangle})$, one recovers

$$V^p_{()}(x)V^q_{()}(y) = \frac{1}{|x-y|^{pq}}(V^{p+q}_{()}(y) + \mathrm{i}p(x-y)^i V^{p+q}_{(i)}(y) + \mathrm{O}((x-y)^2)).$$

3. Schwinger–Dyson morphisms

Given a contraction algebra L, and a solution $\langle\cdot\rangle$ of the Schwinger–Dyson equation, it may happen that there are higher-order elements r in $Sym(L)$ that behave in a way similar to those of L in the sense that there is an operation $[r_i \triangleright r_j]$ such that $\langle r_1 \ldots r_n\rangle = \sum_j \langle r_2..[r_1 \triangleright r_j]..r_n\rangle$. This section will be concerned with this situation.

3.1. Definitions and theorems on Schwinger–Dyson morphisms

Definition 8. Let L_1 and L_2 be two contraction algebras. Then by a Schwinger–Dyson morphism we mean a map $M : L_1 \to Sym(L_2)$ (extended by $M(XY) = M(X)M(Y)$) such that if a linear map $\langle\cdot\rangle_2 : Sym(L_2) \to K$ satisfies the Schwinger–Dyson equation for L_2 then the map $\langle\cdot\rangle_1 := \langle M(\cdot)\rangle_2$ satisfies the Schwinger–Dyson equation for L_1.

A contraction algebra L is said to have a free field realization iff there exists an injective Schwinger–Dyson morphism from L to a Gaussian contraction algebra.

The reader who is familiar with conformal field theory may wish to keep in mind the Sugawara map

$$M(T(z)) := \frac{1}{2k+2c^{\vee}} N(J(T^a, z)J(T_a, z)),$$

in which we will look form our point of view in a moment. In the next theorem we will derive a criterion to check whether a map is Schwinger–Dyson morphism. One may think of E as the Virasoro algebra and of L as the Kac–Moody algebra.

Definition 9. By a sum of two contraction algebras L_1 and L_2, we mean a contraction algebra structure on the vectorspace $L_1 \oplus L_2$ such that the restrictions of the contraction to either component reduce to the original contractions on L_1 and L_2. By an extension of a contraction algebra L we mean a sum $E \oplus L$ such that $[E \triangleright L] \subset L$ and $[L \triangleright E] \subset L$.

The reader may check that the following defines an extension of the Kac–Moody algebra with derivatives $\partial^k J$ by the Virasoro algebra:

$$[J(X,z) \triangleright T(\zeta)] = \frac{J(X,\zeta)}{(z-\zeta)^2}; \qquad [T(z) \triangleright J(X,\zeta)] = \frac{J(X,\zeta)}{(z-\zeta)^2} + \frac{\partial J(X,\zeta)}{z-\zeta}.$$

Theorem 10. *Let $E \oplus L$ be an extension of a sub-Gaussian contraction algebra L, and let $M : E \oplus L \to Sym(L)$ satisfy:*

(i) *$M|_L = id$ (in what follows we will write $M(X) = X$ for $X \in L$);*

(ii) *$\forall_{e\in E} \langle M(e)\rangle_L = 0$ for $\langle\cdot\rangle_L$ a solution of the Schwinger–Dyson equation;*

(iii) *$\forall_{a\in E\oplus L; e\in E} M([a \triangleright e]) = [a \triangleright M(e)]$.*

Then M is a Schwinger–Dyson morphism.

Proof. First note that since $M|_L = id$ we can strengthen the last property of M to $e \in E\oplus L$, and by the derivation property of $[\cdot \triangleright \cdot]$ to $e \in Sym(E \oplus L)$. Now let $\langle\cdot\rangle$ be a solution of the Schwinger–Dyson equation for L. We have to prove that $\forall_{a\in E\oplus L; s\in Sym(E\oplus L)} \langle M(a)M(s)\rangle = \langle M([a \triangleright s])\rangle$. If $a \in L$, we have LHS $= \langle aM(s)\rangle = \langle [a \triangleright M(s)]\rangle = \langle M([a \triangleright s])\rangle$, by the strengthened property of M. It remains to prove the identity for $a = e \in E$. We will first prove that for $s \in Sym(L)$ we have $\langle M(e)s\rangle = \langle [e \triangleright s]\rangle$. (Remember $[e \triangleright s] \in Sym(L)$ because of the definition of an extension). We will prove this by induction on the degree of s. If $s = 1$, this follows from $\langle M(e)\rangle = 0$. So let us assume that the identity was proved for s. We will prove it for Xs with $X \in L$:

$$\begin{aligned}\langle M(e)Xs\rangle &= \langle [X \triangleright M(e)]s\rangle + \langle M(e)[X \triangleright s]\rangle \\ &= \langle M([X \triangleright e])s\rangle + \langle M(e)[X \triangleright s]\rangle.\end{aligned}$$

Using that $[X \triangleright e] \in L$ and $[X \triangleright s]$ has degree lower or equal to s since L is sub-Gaussian so that we can use induction, this equals:

$$\begin{aligned}&\langle [[X \triangleright e] \triangleright s] + [e \triangleright [X \triangleright s]]\rangle \\ &\quad = \langle [[e \triangleright X] \triangleright s] + [X \triangleright [e \triangleright s]]\rangle\end{aligned}$$

$$= \langle [e \triangleright X]s + X[e \triangleright s] \rangle = \langle [e \triangleright Xs] \rangle.$$

We have now proved the identity for Xs, and therefore for $s \in Sym(L)$. It remains to prove it for $s \in Sym(E \oplus L)$. Indeed this follows from the fact that for $s \in Sym(E \oplus L)$ we still have $M(s) \in Sym(L)$ for which we may use our weaker result so that $\langle M(e)M(s) \rangle = \langle [e \triangleright M(s)] \rangle = \langle M([e \triangleright s]) \rangle$, which concludes the proof. □

The next theorem is the contraction-algebraic analogue of the Knizhnik–Zamolodchikov equation.

Theorem 11. *Consider two Schwinger–Dyson morphisms obtained from extensions as in Theorem 10: $M_{13} : L_1 \oplus L_3 \to Sym(L_3)$, and $M_{23} : L_2 \oplus L_3 \to Sym(L_3)$. Let $T \in L_1$ be such that $M_{13}(T) = g_{ab} M_{23}(N_2(J^a J^b))$, where the J^a are in L_2, g_{ab} is a symmetric matrix, and N is normal ordering. Then for $\phi_i \in L_3$:*

$$\sum_i \langle \phi_1 .. ([T \triangleright \phi_i] - g_{ab}[J^a \triangleright [J^b \triangleright \phi_i]]) .. \phi_n \rangle$$
$$= \sum_{i \neq j} g_{ab} \langle \phi_1 .. [J^a \triangleright \phi_i] .. [J^b \triangleright \phi_j] .. \phi_n \rangle.$$

Proof.

$$\sum_i \langle \phi_1 .. [T \triangleright \phi_i] .. \phi_n \rangle$$
$$= \langle M_{13}(T) \phi_1 .. \phi_n \rangle = \langle M_{23}(J^a J_a - [J^a \triangleright J_a]) \phi_1 .. \phi_n \rangle$$
$$= \sum_i \langle M_{23}(J_a) \phi_1 .. [J_a \triangleright \phi_i] .. \phi_n \rangle$$
$$= \sum_i \langle M_{23}(J_a) \phi_1 .. [J_a \triangleright \phi_i] .. \phi_n \rangle + \sum_i g_{ab} \langle \phi_1 .. [J^a \triangleright [J^b \triangleright \phi_i] .. \phi_n \rangle. \qquad \square$$

To recover the usual KZ equation, take L_1 to be the Virasoro algebra, L_2 the Kac–Moody algebra of a Lie algebra L, take the quadratic relation between $T(z)$'s and $J(z)$'s to be given by the Sugawara construction, and assume L_3 is a module for both Virasoro and Kac–Moody in the sense of Definition 7, i.e with contractions

$$[T(z) \triangleright \phi(v, \zeta)] := \frac{h.\phi(v, \zeta)}{(z - \zeta)^2} + \frac{\partial \phi(v, \zeta)}{(z - \zeta)}; \qquad [J(X, z) \triangleright \phi(v, \zeta)] := \frac{\phi(Xv, \zeta)}{z - \zeta}.$$

The Knizhnik–Zamolodchikov equation now reads:

$$\sum_i \langle \phi(v_1, z_1) .. ([T(\zeta) \triangleright \phi(v_i, z_i)]$$
$$- \frac{1}{2k + 2c^\vee} [J^a(\zeta) \triangleright [J_a(\zeta) \triangleright \phi(v_i, z_i)]]) .. \phi(v_n, z_n) \rangle$$
$$= \sum_{i \neq j} \frac{1}{2k + 2c^\vee} \langle \phi(v_1, z_1) .. [J^a \triangleright \phi(v_i, z_i)] .. [J_a \triangleright \phi(v_j, z_j)] .. \phi(v_n, z_n) \rangle.$$

Making $\oint_{z_k} \mathrm{d}\zeta$ act on this equation then gives the usual form

$$\frac{\partial}{\partial z_k}\left\langle \prod_i \phi(v_i, z_i)\right\rangle$$
$$= \frac{1}{k+c^{\vee}} \sum_{i\neq k} \frac{1}{z_i - z_k} \langle \phi(v_1, z_1)..\phi(T^a v_i, z_i)..\phi(T_a v_k, z_k)..\phi(v_n, z_n)\rangle.$$

3.2. Details of the Sugawara calculation using sub-Gaussian calculus

In this section we will prove that the Sugawara construction gives a Schwinger–Dyson morphism. The calculation is the contraction-algebraic analogue of the usual Sugawara construction in operator language.

Theorem 12. *Let L be a finite-dimensional reductive Lie algebra with invariant metric h and basis $\{T_a\}$ such that $ad(T_aT^a) = 2c^{\vee} id_L$. For $k \in \mathbb{R}$ define the Sugawara map from the Virasoro algebra to the Kac–Moody algebra of (L, kh) as follows:* [6]

$$M(T(Z)) := \frac{1}{2k+2c^{\vee}} N(J(T^a, z)J(T_a, z)).$$

If $c = [k \dim(L)/(k + c^{\vee})]$ then the Sugawara map is a Schwinger–Dyson morphism.

Proof. We will check that
(i) $[J(X, z) \rhd M(T(\zeta))] = M([J(X, z) \rhd T(\zeta)])$,
(ii) $[T(z) \rhd M(T(\zeta))] = M([T(z) \rhd T(\zeta)])$.
Indeed, using that $ad(T_aT^a) = 2c^{\vee} id_L$ and

$$[T_a, T^b]^c T_b = h(T^c, [T_a, T^b])T_b = h([T^c, T_a], T^b)T_b = [T^c, T_a],$$

and using Theorem 2(iv), we get

$$(2k + 2c^{\vee})[J_a(z) \rhd M(T(\zeta))]$$
$$= [J_a(z) \rhd N(J^b(\zeta)J_b(\zeta))]$$
$$= N([J_a(z) \rhd J^b(\zeta)]J_b(\zeta) + J^b(\zeta)[J_a(z) \rhd J_b(\zeta)])$$
$$+ \tfrac{1}{2}N([[J^b(\zeta) \rhd J_a(z)] \rhd J_b(\zeta)] + [[J_b(\zeta) \rhd J_a(z)] \rhd J^b(\zeta)])$$
$$= N\left(2\left\{\frac{kh_{ab}}{(z-\zeta)^2} + \frac{J([T_a, T_b], \zeta)}{z-\zeta}\right\} J^b(\zeta) + \left[\frac{J([T_b, T_a], z)}{\zeta - z} \rhd J^b(\zeta)\right]\right)$$
$$= \frac{2kJ_a(\zeta)}{(z-\zeta)^2} + A + B,$$

[6] This needs some explanation since $[J(z) \rhd J(z)]$ is undefined. We will not use any specific value for this contraction however: We will only use formula iv of Theorem 2. This is really a theorem about pre Lie algebras P if we define normal ordering to be $Sym(P) \to Sym(P)$. Thus, to define the above calculus with undefined contractions, we go to the universal pre Lie algebra on symbols $J(X, z)$, 1, and impose the relation $[J(X, z) \rhd J(Y, w)] = h(X, Y)/(z-w)^2 + J([X, Y], w)/(z-w)$ *only* for $z \neq w$. Thus, $[J(X, z) \rhd J(Y, z)]$ remains a symbol.

where

$$(z-\zeta)A = 2N(J([T_a, T_b], \zeta)J^b(\zeta)) = 2N(J(T_c, \zeta)J([T_a, T_b]^c T^b, \zeta))$$
$$= 2N(J_c(\zeta)J([T^c, T_a], \zeta)) = -A(z-\zeta) = 0,$$

and

$$B = \frac{1}{z-\zeta}\left\{\frac{h([T_a, T_b], T^b}{(z-\zeta)^2} + \frac{J([[T_a, T_b], T^b], \zeta)}{z-\zeta}\right\} = 0 + \frac{2c^\vee J(T_a, \zeta)}{(z-\zeta)^2}.$$

As for the second point:

$$\begin{aligned}
&[T(z) \triangleright N(J_a(\zeta)J^a(\zeta))] \\
&= N(2[T(z) \triangleright J_a(\zeta)]J^a(\zeta) + [[J_a(\zeta) \triangleright T(z)] \triangleright J^a(\zeta)]) \\
&= N\left(\frac{2J_a(\zeta)J^a(\zeta)}{(z-\zeta)^2} + \frac{2\partial J_a(\zeta)J^a(\zeta)}{z-\zeta} + \left[\frac{J_a(z)}{(\zeta-z)^2} \triangleright J^a(\zeta)\right]\right) \\
&= \frac{2}{(z-\zeta)^2}N(J_a(\zeta)J^a(\zeta)) + \frac{1}{z-\zeta}\partial N(J_a(\zeta)J^a(\zeta)) \\
&\quad + \frac{1}{(\zeta-z)^2}\left\{\frac{kh_{ab}h^{ab}}{(z-\zeta)^2} + \frac{J([T^a, T_a], \zeta)}{z-\zeta}\right\} \\
&= (2k+2c^\vee)\left\{\frac{2T(\zeta)}{(z-\zeta)^2} + \frac{\partial T(\zeta)}{z-\zeta}\right\} + \frac{k\dim(L)}{(z-\zeta)^4}.
\end{aligned}$$

This proves that the Sugawara construction leads to a Schwinger–Dyson morphism. □

3.3. Remarks on higher-dimensional morphisms

We will now make some general remarks on why it is difficult to find analogous Schwinger–Dyson morphisms in higher dimension, i.e. where for every $x \in \mathbb{R}^n$ there is a $T(x) \in Sym(L)$ with properties analogous to $T(z)$ above.

Recall that a Lagrangian $\mathcal{L}$, when given a classical field γ, produces a volume form $\mathcal{L}(\gamma)$ on some manifold M which can then be integrated to get the action $S(\gamma)$. That given such a Lagrangian, to every vector field X on M is associated its Noether current $J(X)$, where $J(X)(\gamma) \in \Omega^{|M|-1}(M)$ can be integrated over $(|M|-1)$-submanifolds to give the charge $Q(X) := \int J(X)$. Finally that Noether's theorem states that if X is a symmetry of the original Lagrangian, then $\mathrm{d}J(X)$ is a conserved current in the sense that it can be written as a multiple of the functional derivative $\delta\mathcal{L}/\delta\gamma$, so that if γ is a solution of the Euler–Lagrange equation $\delta\mathcal{L}/\delta\gamma = 0$, then $\mathrm{d}J(X)(\gamma) = 0$.

By putting together the notion of a conserved current and the Schwinger–Dyson equation (where we now take L to be spanned by the functional derivatives $\delta/\delta\gamma(m)$), and assuming that we are dealing with actions as before where normal ordering N makes sense, we obtain the Ward identity.

Theorem 13 (Ward identity). *Let J be a conserved current for $\mathcal{L}$, defined on the base manifold M except for a few singular points $s_i \in M$. Let S_i be $(|M|-1)$ spheres surrounding the s_i. For $x_j \in M$, let $\mathcal{O}(x_j)$ be functionals of the classical field γ such that if $m \neq n$, then $\delta\mathcal{O}(m)/\delta\gamma(n) = 0$. Further, let $Q := \sum_i \int_{S_i} N(J)$, where N denotes normal ordering. Then*

$$\left\langle Q \prod_j \mathcal{O}(x_j) \right\rangle = \sum_j \left\langle [Q \triangleright \mathcal{O}(x_j)] \prod_{j' \neq j} \mathcal{O}(x_{j'}) \right\rangle,$$

where the "contraction" $[Q \triangleright \cdot]$ is defined by multiplication

$$[Q \triangleright \mathcal{O}(x_j)] := \int_{S_j} N(J)\mathcal{O}(x_j),$$

and S_j is a sphere that surrounds x_j.

Proof. Let $\tilde{M}$ be M with balls B_i and B_j removed around the singularities s_i of J and the insertions x_j. The difference between the left-hand side and the right-hand side reads:

$$\int_{\partial\tilde{M}} \left\langle N(J) \prod_j \mathcal{O}(x_j) \right\rangle = \int_{\tilde{M}} \left\langle N(\mathrm{d}J) \prod_j \mathcal{O}(x_j) \right\rangle.$$

Since J is conserved, there is an $\mathcal{R}$ such that $\mathrm{d}J|_m = \mathcal{R}(m)(\delta\mathcal{L}/\delta\gamma(m))$, so that:

$$= \int_{\tilde{M}} \left\langle N\left(\mathcal{R}\frac{\delta\mathcal{L}}{\delta\gamma(\tilde{m})}\right) \prod_j \mathcal{O}(x_j) \right\rangle = \int_{\tilde{M}} \left\langle \mathcal{R}\frac{\delta}{\delta\gamma(\tilde{m})} \prod_j \mathcal{O}(x_j) \right\rangle = 0,$$

since $x_j \notin \tilde{M}$. □

From the above and recalling the identity $\langle rs \rangle = \langle [r \triangleright s] \rangle$ holding if r and s are in the image of a Schwinger–Dyson morphism, it is obvious that normal ordered changes of Noether symmetries are good candidates to be in such images. Indeed, the examples of two-dimensional conformal field theory are of this type: Since the vector field $X_{(w)} := \{z \mapsto (1/(z-w))\partial_z\}$ is a (complexified) conformal Killing field, every two-dimensional conformal field theory allows it locally as a symmetry. The consequence is that using Cauchy's theorem, the Noether current density T_{zz} defined by $J(X) = T_{zz}X^z\,\mathrm{d}z + T_{\bar{z}\bar{z}}X^{\bar{z}}\,\mathrm{d}\bar{z}$ can be written as a charge:

$$T_{zz}(w) = \frac{1}{2\pi i}\oint_w \frac{T_{zz}(z)\,\mathrm{d}z}{z-w} = \oint_w J\left(\frac{X_{(w)}}{2\pi i}\right) = Q\left(\frac{X_{(w)}}{2\pi i}\right).$$

Thus, the Ward identity which holds in general for *charges* now also holds for the current *density* at a point. This is an aspect which is difficult to realize in higher dimensions: Indeed, having symmetry vector fields with poles together with translational symmetry implies an infinite-dimensional symmetry algebra since vector fields with poles at different locations are linearly independent. Thus, Schwinger–Dyson morphisms of the form $T(x) = N(J(x)J(x))$ are not to be expected in higher dimension without having infinite-dimensional symmetry.

Note that this, however, is exactly what Johnson and Low [8] tried to do: From their investigation of the Jordan construction $J(x) = N(\psi(x)\psi(x))$ in higher dimension, they found expressions for the commutators of the $J(x)$'s which did not satisfy the Jacobi identity. The analogoue in our language is that one does not find an operation $[J(x) \triangleright J(y)]$ which satisfies the pre-Lie property as in the two-dimensional case. Note the advantage of our approach that it is not logically incorrect not to find such an operation, whereas it is logically incorrect to claim to have operators of which the commutators do not satisfy the Jacobi identity.

4. An existence theorem for solutions of the Schwinger–Dyson equation

Theorem 14. *If normal ordering N is invertible, then the Schwinger–Dyson has a unique solution up to normalization $\langle\cdot\rangle = ZN^{-1}$, where $Z : Sym(L) \to K$ is the projection on scalars.*

In this section we will be concerned with proving Theorem 14 for general contractions $L \otimes L \to Sym(L)$ with possibly non-Abelian L. Our first aim will be to prove that the solution is necessarily given by ZN^{-1}.

Proof of Theorem 14 (Uniqueness). Let I satisfy Schwinger–Dyson equation. We will prove that $\forall_{s\in Sym(L)} IN(s) = Z(s)I(1)$. So if N is invertible, then $I = I(1).ZN^{-1}$: For $\deg(s) = 0$, this equation reads $sI(1) = sI(1)$, and for higher degree, we have:

$$
\begin{aligned}
IN(X_1 \dots X_n) = \frac{1}{n} I \sum_{i=1}^{n} X_i N(X_1..\hat{X}_i..X_n) \\
-[X_i \triangleright N(X_1..\hat{X}_i..X_n)] = 0 = Z(X_1..X_n). \qquad \square
\end{aligned}
$$

Thus, to prove existence it suffices to prove that ZN^{-1} satisfies the Schwinger–Dyson equation, i.e. we have to prove that $ZN^{-1}(Xs - [Xs \triangleright s]) = 0$. There will be some complications which we will first illustrate in the easiest case $s = Y \in L$: we then have to prove that $ZN^{-1}(XY - [X \triangleright Y]) = 0$. Now

$$
XY - [X \triangleright Y] = N(XY) + \tfrac{1}{2}[X \triangleright Y] + \tfrac{1}{2}[Y \triangleright X] - [X \triangleright Y] = N\big(XY + \tfrac{1}{2}[Y, X]\big)
$$

so that indeed $ZN^{-1}(XY - [X \triangleright Y]) = 0$. We have to generalize this procedure to arbitrary s. Our first step will be to prove the formula for $N(s)$ instead of s, which is the same if N is invertible: i.e. we will prove $ZN^{-1}(XN(s) - [X \triangleright N(s)]) = 0$. This version is better suited for proof by induction, in view of the definition of normal ordering. So the question is: Given X and s, can we construct $R(X, s)$ such that $XN(s) - [X \triangleright N(s)] = N(R)$ and

$Z(R) = 0$? What we see above is that $R(X, Y) = XY + \frac{1}{2}[Y, X]$. In Theorm 19 we will prove that $\nu = N(M+r)$, where $\nu(X, s) := XN(s) - [X \triangleright N(s)]$, and M and r are known, i.e. we can take $R := M + r$, and the theorem will be proved.

4.1. Preliminaries on the symmetric algebra of a Lie algebra L

We will be defining maps on $Sym(L)$ without going through the explicit symmetrization every time. To that end we include the following theorem. It is, say, the statement that in symmetric algebras every element of $Sym^n(V)$ can be written as a sum of nth powers, for example $2XY = (X+Y)^2 - X^2 - Y^2$. This will simplify matters when proving that $ZN^{-1}(XN(s) - [X \triangleright N(s)) = 0$, because we will only prove that $ZN^{-1}(XN(Y^n) - [X \triangleright N(Y^n)]) = 0$, which as we will see is easier.

Theorem 15 (Polarization). *Let V, W be vectorspace, $G : V^{\otimes n} \to W$ linear, then there is a unique linear map $G_s : Sym^n(V) \to W$ such that $\forall_{v \in V} G_s(v^n) = G(v^n)$.*

Proof. Existence is evident from the following example:

$$G_s(X_1..X_n) := \frac{1}{n!} \sum_{\sigma \in S_n} G(X_{\sigma(1)}, ..., X_{\sigma(n)}).$$

Next, we have the following formula in $Sym(V)$:

$$n! X_1 \ldots X_n = \sum_{S \subset \{1,\ldots,n\}} (-1)^{n-|S|} \left(\sum_{s \in S} X_s \right)^n,$$

which is proved by noting that both sides are symmetric and homogeneous polynomials which can be divided by X_1, so that both sides are equal up to scalar multiplication. To determine this factor, we take $X_1 = X_2 = \cdots = X_n$, and use that $\sum_{k=0}^{n} (-1)^{n-k} k^n \binom{n}{k} = n!$ This gives uniqueness, since

$$G_s(X_1..X_n) = \frac{1}{n!} \sum_{S \subset \{1,\ldots,n\}} (-1)^{n-|S|G_s} \left(\sum_{s \in S} X_s \right)^n$$
$$= \frac{1}{n} \sum_{S \subset \{1,\ldots,n\}} (-1)^{n-|S|G} \left(\sum_{s \in S} X_s \right)^n. \qquad \square$$

Definition 16. Let L be a Lie algebra. Define the following maps:

(i) $Z : Sym(L) \to K \leq Sym(L)$, "zero degree projection".

(ii) $M : L \otimes Sym(L) \to Sym(L)$; "multiply":

$$M(X \otimes s) := Xs.$$

(iii) $S : Sym(L) \to L \otimes Sym(L)$; "split":

$$S(1) := 0; \quad S(X_1 \ldots X_n) := \frac{1}{n} \sum_{i=1}^{n} X_i \otimes X_1..\hat{X}_i..X_n.$$

(iv) $\Sigma : L \otimes Sym(L) \to L \otimes Sym(L)$; "symmetrize":

$$\Sigma(X_0 \otimes X_1..X_n) := \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} X_{\sigma(0)} \otimes X_{\sigma(1)}..X_{\sigma(n)}.$$

(v) $r := \oplus_n r_n : L \otimes Sym^n(L) \to Sym^{[1,n]}(L)$: "rest term", and $C := \bigoplus_n C_n : L \otimes Sym^n(L) \to L \otimes Sym^{[0,n-1]}(L)$ "commutator term", inductively as follows: [7]

$$r(X, 1) := 0; \qquad C(X, 1) := 0,$$
$$C(X, Y^{n+1}) := \frac{n+1}{n+2}\{[Y, X] \otimes Y^n + Y \otimes r(X, Y^n)\}.$$
$$r(X, Y^{n+1})$$
$$:= \frac{n+1}{n+2}\{[Y, X]Y^n + Yr(X, Y^n) + r([Y, X], Y^n) + r(Y, r(X, Y^n))\}.$$

(vi) $M_\lambda := M + \lambda r$; "modified multiplication".

(vii) $\pi_\lambda := SM_\lambda$, "projection". [8]

Theorem 17. *These maps satisfy the following properties*:

(i) $M_\lambda = M + \lambda r, \pi_\lambda = SM_\lambda$,

(ii) $r = (M + r)C$,

(iii) $\Sigma^2 = \Sigma$,

(iv) $\Sigma = SM = \pi_0$,

(v) $Zr = ZM = 0$,

(vi) $CS = 0, rS = 0$,

(vii) $\pi_\lambda \Sigma = \Sigma$,

(viii) $\pi_\lambda = \lambda\Sigma + \pi_\lambda C$,

(ix) $\pi_\lambda^2 = \pi_\lambda$.

Proof.

(i) By definition.

(ii)
$$\begin{aligned} r(X, Y^{n+1}) &= \frac{n+1}{n+2}\{[Y, X]Y^n + Yr(X, Y^n) + r([Y, X], Y^n) \\ &\qquad + r(Y, r(X, Y^n))\} \\ &= \frac{n+1}{n+2}(M + r)\{[Y, X] \otimes Y^n + Y \otimes r(X, Y^n)\} \\ &= (M + r)C(X, Y^{n+1}). \end{aligned}$$

(iii) Left to the reader.

[7] The motivation for the definition of r comes from the proof of Lemma 20: It is chosen in such a way that the fifth equality of the proof holds. The definition of C is useful since we then have $r = (M + r)C$, as proved in Theorem 17.

[8] This map with $\lambda = 1$ is used in Theorem 23. In that theorem a condition of the form $J = J\pi$ appears, which motivates us to try to prove that $\pi^2 = \pi$. This is indeed the case, as is demonstrated in Theorem 17.

(iv) By polarization it suffices to prove that $\Sigma(X \otimes Y^n) = SM(X \otimes Y^n)$. Indeed:

$$\mathrm{LHS} = \frac{1}{(n+1)}(X \otimes Y^n + nY \otimes XY^{n-1}) = S(XY^n) = \mathrm{RHS}.$$

(v) $ZM(X \otimes s) = Z(Xs) = 0$. Next, we prove by induction on $|s|$ that $Zr(X, s) = 0$. Indeed, $Zr(X, 1) = Z(0) = 0$; suppose that the identity holds up to degree n. Then we have

$$Zr(X, Y^{n+1}) = Z(M + r)C(X, Y^{n+1}) = (Zr)C(X, Y^{n+1}) = 0$$

by induction since C lowers degree.

(vi) We will prove by induction on $|s|$ that $CS(s) = 0$ and $rS(s) = 0$. Indeed $CS(1) = C(0) = 0$, $rS(1) = r(0) = 0$, $CS(X) = C(X \otimes 1) = 0$, and $rS(X) = r(X \otimes 1) = 0$. So assume these identities hold up to degree n+1. Then

$$\begin{aligned} CS(X^{n+2}) &= C(X \otimes X^{n+1}) = \frac{n+1}{n+2} X \otimes r(X, X^n) \\ &= \frac{n+1}{n+2} X \otimes rS(X^{n+1}) = 0 \end{aligned}$$

and

$$rS(X^{n+2}) = (M + r)CS(X^{n+2}) = 0.$$

(vii) $\pi_\lambda \Sigma = S(M + \lambda r)\Sigma = \Sigma^2 + \lambda SrSM = \Sigma^2 = \Sigma$.

(viii) $\pi_\lambda = S(M + \lambda r) = SM + \lambda S(M + r)C = \lambda SM + S(M + \lambda r)C = \lambda\Sigma + \pi_\lambda C$.

(ix) We will prove by induction on $|s|$ that $\pi_\lambda^2(X \otimes s) = \pi_\lambda(X \otimes s)$. Indeed, $\pi_\lambda(X \otimes 1) = X \otimes 1 = \pi_\lambda^2(X \otimes 1)$. Next, assume that the identity holds up to degree n, and that $|s| = n + 1$. Then since C lowers degree, we have $\pi_\lambda^2 C(X \otimes s) = \pi_\lambda C(X \otimes s)$, so that

$$\begin{aligned} \pi_\lambda^2(X \otimes s) &= \pi_\lambda(\lambda\Sigma + \pi_\lambda C)(X \otimes s) \\ &= \lambda\pi_\lambda \Sigma(X \otimes s) + \pi_\lambda^2 C(X \otimes s) \\ &= \lambda\Sigma(X \otimes s) + \pi_\lambda C(X \otimes s) \\ &= (\lambda\Sigma + \pi_\lambda C)(X \otimes s) = \pi_\lambda(X \otimes s). \end{aligned}$$

This completes the proof. □

4.2. *Theorems involving contractions*

Definition 18. Given a polynomial contraction on L, we define the following maps: $N : Sym(L) \to Sym(L)$, "normal ordering", and $\nu : L \otimes Sym(L) \to Sym(L)$, inductively as follows: $N(1) := 1$, and

$$\begin{aligned} N(X_1 \ldots X_n) &:= \frac{1}{n} \sum_{i=1}^{n} X_i N(X_1..\hat{X}_i..X_n) - [X_i \rhd N(X_1..\hat{X}_i..X_n)], \\ \nu(X \otimes s) &:= XN(s) - [X \rhd N(s)]. \end{aligned}$$

The idea underlying the following theorem is as follows: Recall that we want to prove that there is an R such that $\nu(X \otimes s) = N(R(X \otimes s))$ and $ZR = 0$. In other words, we have to rewrite $\nu(X \otimes Y^n)$ as the N of something, which we are going to try by induction on n. So given the fact that there is an $r(X, Y^n)$ such that

$$\nu(X, Y^n) = N(XY^n + r(X, Y^n)),$$

we want to construct $r(X, Y^{n+1})$ such that it satisfies this equation with n replaced by $n+1$. In the course of Lemma 20 we find that $\nu(X, Y^{n+1})$ equals

$$\nu(Y, XY^n) + N(Yr(X, Y^n) + r(Y, r(X, Y^n)) + [Y, X]Y^n + r([Y, X], Y^n)).$$

So at that stage we have expressed $\nu(X, Y^{n+1})$ in terms of the N something and $\nu(Y, XY^n)$. So it remains to express $\nu(Y, XY^n)$ in terms of $N(..)$ and $\nu(X, Y^{n+1})$ in an independent way, so that we get two equations

$$a\nu(X, Y^{n+1}) + b\nu(Y, XY^n) = N(..), \qquad c\nu(X, Y^{n+1}) + d\nu(Y, XY^n) = N(..),$$

which we may solve for $\nu(X, Y^{n+1})$. This second equation is furnished by Theorem 19(ii), which makes the proof possible. The map $R = M + r$ thus defined indeed satisfies $ZR = 0$, by definition.

Theorem 19. *These maps satisfy the following properties:*
(i) $N = \nu S + Z$,
(ii) $\nu(Y, XY^n) = (-1/(n+1))\nu(X, Y^{n+1}) + N((n+2)/(n+1))XY^{n+1})$,
(iii) $\nu(X, Y^{n+1}) = Y.\nu(X, Y^n) - [Y \triangleright \nu(X, Y^n)] + \nu([Y, X], Y^n)$,
(iv) $\nu = N(M + r)$ *(this is the main result).*

Proof.
(i) Since $S(1) = 0$, we have $N(1) = (\nu S + Z)(1)$, and for higher degree, we have to check $N(X_1..X_n) = \nu S(X_1..X_n)$, which is true by definition of S, ν and N.

(ii)
$$N(XY^{n+1}) = \nu S(XY^{n+1}) = \frac{1}{n+2}(\nu(X, Y^{n+1}) + (n+1)\nu(Y, XY^n))$$
$$\Rightarrow (n+2)N(XY^{n+1}) - \nu(XY^{n+1}) = (n+1)\nu(Y, XY^n).$$

(iii)
$$\begin{aligned}\text{LHS} &= XN(Y^{n+1}) - [X \triangleright N(Y^{n+1})]\\ &= XYN(Y^n) - X[Y \triangleright N(Y^n)] - [X \triangleright YN(Y^n)] + [X \triangleright [Y \triangleright N(Y^n)]]\\ &= Y\nu(X, Y^n) - [Y \triangleright XN(Y^n)] - [X \triangleright Y]N(Y^n) + [[X, Y] \triangleright N(Y^n)]\\ &\quad + Y[X \triangleright N(Y^n)] + [Y \triangleright X]N(Y^n) - Y[X \triangleright N(Y^n)] + [Y \triangleright [X \triangleright N(Y^n)]]\\ &= Y\nu(X, Y^n) - [Y \triangleright \nu(X, Y^n)] + [Y, X]N(Y^n) - [[Y, X] \triangleright N(Y^n)]\\ &= \text{RHS}.\end{aligned}$$

(iv) We prove that $\nu(X \otimes s) = N(M + r)(X \otimes s)$ by induction on $n = |s|.n = 0$: $\nu(X, 1) = X = N(X) = N(X.1 + r(X, 1)) = N(M + r)(X, 1)$. Assume true up to n. Then by polarization it suffices to prove the following.

Lemma 20. $\nu(X, Y^{n+1}) = N(XY^{n+1} + r(X, Y^{n+1}))$.

Proof.

$$\begin{aligned}
\nu(X, Y^{n+1}) &= Y\nu(X, Y^n) - [Y \triangleright \nu(X, Y^n)] + \nu([Y, X], Y^n) \\
&= YN(XY^n + r(X, Y^n)) - [Y \triangleright N(XY^n + r(X, Y^n))] \\
&\quad + N([Y, X]Y^n + r([Y, X], Y^n)) \\
&= \nu(Y, XY^n) + \nu(Y, r(X, Y^n)) + N([Y, X]Y^n + r([Y, X], Y^n)) \\
&= \nu(Y, XY^n) + N(Yr(X, Y^n) + r(Y, r(X, Y^n)) + [Y, X]Y^n \\
&\quad + r([Y, X], Y^n)) \\
&= \nu(Y, XY^n) + N\left(\frac{n+2}{n+1} r(X, Y^{n+1})\right) \\
&= \frac{-1}{n+1}\nu(X, Y^{n+1}) + N\left(\frac{n+2}{n+1} XY^{n+1} + \frac{n+2}{n+1} r(X, Y^{n+1})\right) \\
&\quad \Rightarrow \nu(X, Y^{n+1}) = N(XY^{n+1} + r(X, Y^{n+1})).
\end{aligned}$$

□

4.3. Applications to sub-Gaussian algebras and boundary terms

In the rest of this section we will apply some of the above fromulae to explicitly construct the inverse of normal ordering in the sub-Gaussian case, to give a generalization of the sub-Gaussian formula for $[X \triangleright N(YZ)]$, proved in Theorem 2(iv), and to construct the solution of the Schwinger–Dyson equation with prescribed boundary term.

Definition 21. Given a sub-Gaussian contraction algebra, we define $\bar{N} : Sym(L) \to Sym(L)$, and $\rho : (K \oplus L) \otimes Sym(L) \to Sym(L)$, inductively by $(r(1, s) := 0)$.

$$\begin{aligned}
&\bar{N}(1) := 1, \\
&\bar{N}(Y^{n+1}) := Y\bar{N}(Y^n) + r(Y, \bar{N}(Y^n)) + \bar{N}([Y \triangleright Y^n]). \\
&\rho(1, s) := \rho(X, 1) := 0, \\
&\rho(X, Y^{n+1}) := [[Y \triangleright X] \triangleright Y^n] + \rho([Y \triangleright X], Y^n) + Y\rho(X, Y^n) \\
&\qquad + r(Y, \rho(X, Y^n)) + r([X \triangleright Y], Y^n) + r(Y, [X \triangleright Y^n]).
\end{aligned}$$

Theorem 22. *We then have*:
(i) $[X \triangleright N(Y^n)] = N([X \triangleright Y^n] + \rho(X, Y^n))$,
(ii) $N^{-1} = \bar{N}$.

Proof.
(i) We will prove this identity by induction on n. For $n = 0$ it reads $0 = 0$, so assume it to be true up to n, we will now prove it for $n + 1$ (subscripts indicate corresponding terms):

$$[X \triangleright N(Y^{n+1})] - N([X \triangleright Y^{n+1}])$$

$$\begin{aligned}
&= [X \rhd YN(Y^n)] - [X \rhd [Y \rhd N(Y^n)]] - N([X \rhd Y]Y^n) - N(Y[X \rhd Y^n]) \\
&= [X \rhd Y]N(Y^n)_1 - [[X, Y] \rhd N(Y^n)]_2 - [X \rhd Y]N(Y^n)_1 \\
&\quad - YN([X \rhd Y^n])_3 + Y[X \rhd N(Y^n)]_3 - [Y \rhd [X \rhd N(Y^n)]]_4 \\
&\quad + [[X \rhd Y] \rhd N(Y^n)]_2 + [Y \rhd N([X \rhd Y^n])]_4 \\
&\quad + N(r([X \rhd Y], Y^n)) + N(r(Y, [X \rhd Y^n])) \\
&= [[Y \rhd X] \rhd N(Y^n)]_2 + YN(\rho(X, Y^n))_3 - [Y \rhd N(\rho(X, Y^n))]_4 \\
&\quad + N(r([X \rhd Y], Y^n) + r(Y, [X \rhd Y^n])) \\
&= N\{[[Y \rhd X] \rhd Y^n] + \rho([Y \rhd X], Y^n) + Y\rho(X, Y^n) + r(Y, \rho(X, Y^n)) \\
&\quad + r([X \rhd Y], Y^n) + r(Y, [X \rhd Y^n])\} \\
&= N(\rho(X, Y^{n+1})).
\end{aligned}$$

(ii) We will prove by induction on n that $N\bar{N}(Y^n) = \bar{N}N(Y^n) = Y^n$. Indeed, this is true by definition for $n = 0$, so assume the identities hold up to n, then:

$$\begin{aligned}
N\bar{N}(Y^{n+1}) &= N(Y\bar{N}(Y^n) + r(Y, \bar{N}(Y^n))) + [Y \rhd Y^n] \\
&= \nu(Y, \bar{N}(Y^n)) + [Y \rhd Y^n] \\
&= YN\bar{N}(Y^n) - [Y \rhd N\bar{N}(Y^n)] + [Y \rhd Y^n] = Y^{n+1},
\end{aligned}$$

Next, to prove $\bar{N}N = id$, we first prove that $\bar{N}$ is surjective. This follows from $\bar{N}(Y^{n+1}) = Y^{n+1} \bmod Sym^{[0,n]}(L)$, which in turn follows from the definition, by induction on n. Therefore, for every Y^n there is an s_n such that $Y^n = \bar{N}(s_n)$, so that

$$\bar{N}N(Y^n) = \bar{N}N\bar{N}(s_n) = \bar{N}(s_n) = Y^n,$$

which proves the identity. □

The following theorem is motivated by integration over manifolds with boundary: Suppose we already know integration over the boundary ∂M of a manifold. Then in particular, if μ is a volume form on M, we know $\tilde{J} : X \otimes f \mapsto \int_{\partial M} f i_X \mu$. This last association is related to the integral over M through the Schwinger–Dyson equation for $I : f \mapsto \int_M f\mu$:

$$I(\nabla(X)f + X(f)) = \int_M L_X(f\mu) = \int_M di_X(f\mu) = \int_{\partial M} f i_X \mu = \tilde{J}(X \otimes f).$$

In algebraic language, this leads us to consider the equation $I(-Xs + [X \rhd s]) = \tilde{J}(X \otimes s)$, i.e. setting $J(X \otimes s) := -\tilde{J}(X \otimes N(s))$, we become interested in the solvability of the equation $I(XN(s) - [X \rhd N(s)]) = J(X \otimes s)$, which is what the following theorem is about:

Theorem 23. *Setting $\pi := S(M + r)$, the following are equivalent:*
(1) $I(XN(s) - [X \rhd N(s)]) = J(X \otimes s)$,
(2) $IN = JS + I(1)Z$ *and* $J = J\pi$.

Proof. First, using Theorems 17 and 19, with $R := M+r$, we have the following properties: $ZR = 0$, $SZ = 0$, $\nu = \nu SR$, $N = \nu S + Z$. We now prove the theorem. By definition, (1) is equivalent with $I\nu = J$.

(1) $\Rightarrow$ (2): $IN = I\nu S + IZ = JS + I(1)Z$; $J = I\nu = I\nu SR = JSR = J\pi$.

(1) $\Leftarrow$ (2): $I\nu = I\nu SR = I(N-Z)R = INR = JSR + I(1)ZR = J\pi + 0 = J$. □

If N is invertible, then what we have done is to solve the inhomogeneous linear equation $I\nu = J$ as $I = JSN^{-1} + I(1)ZN^{-1}$, which is expected is of the form $I_p + I_0$, where I_p is any solution, and I_0 is the general solution of the homogeneous equation. Further, note that $\pi^2 = \pi$, see Theorem 17.

5. Conclusion

What we have seen is the following:

(i) A useful generalization of contractions to non-Gaussian actions S is the second derivative of S, written as a polynomial in the first derivatives. In view of the fact that normal ordering can be defined in terms of contractions, this leads to a non-Gaussian notion of normal ordering independent of underlying dimension.

(ii) For the Gaussian case, normal ordering is an invertible operation. Invertibility of normal ordering is interesting for non-Gaussian integrals too, since in that case the inverse of normal ordering is directly related to the solution of the Schwinger–Dyson equation. This statement can be generalized to a non-Abelian setting.

(iii) We defined the notion of a sub-Gaussian weight, for which a number of generalized Wick rules can be derived. Examples of sub-Gaussian algebras can be found in two-dimensional conformal field theory.

(iv) We defined the notion of a Schwinger–Dyson morphism which relates solutions of the Schwinger–Dyson equations of different contractions, and formulated a criterion to check whether a map is such a morphism.

Our method has been rather different from the usual approach in that we have not used operators and Hilbert spaces, but rather a purely symbolic approach; the notion of composite operators was now replaced by that Schwinger–Dyson morphisms. In this way we avoided the difficulties of operator-valued distributions and the problems encountered by Johnson and Low in higher-dimensional current algebra, namely that what they call "commutators" of "operators" do not satisfy the Jacobi-identity.

5.1. Positivity

We have not touched upon the analysis of the combined problem of the Schwinger–Dyson equation and positivity, and we will conclude this article with some remarks in that direction.

The main interest of this combined problem lies in a hypothesis that this problem can have at most one solution, the (incomplete) argument being as follows: Let I be a positive solution of the Schwinger–Dyson equation for action S. We wish to communicate the sketch of a proof that at least on $\mathbb{R}^n$, I is determined up to a constant factor. Indeed, assume I is a functional that satisfies the conditions to Riesz' representation theorem. Then there is a measure ν such that $I(f) = \int f\nu$. "Divide" this measure by $\mu = \mathrm{d}x^1 \ldots \mathrm{d}x^n$ to get a positive function, and take the logarithm, so as to get $I(f) = \int f e^{-P}\mu$. Since I now satisfies the Schwinger–Dyson equation for both S and P, we have $\forall_f I(\partial_i(S-P)f) = 0$, which by positivity gives $\partial_i(S) = \partial_i(P)$, or $P = S + c$, so that $I(f) = K . \int f \mathrm{e}^{-S}\mu$, so that I is determined up to a positive scalar.

From this remark we conclude that the essential input data for constructing functional integrals is a combination (S, C) of an action S, and a normalization condition C. Then constructing the functional integral for (S, C) means constructing a positive solution of the Schwinger–Dyson equation of S, satisfying the normalization condition C. This normalization condition may be $\langle 1 \rangle = 1$ but need not always be of that form. For example Gaussian fermionic integrals with singular operator A in the action have $\langle 1 \rangle = 0$ by the Schwinger–Dyson equation which is obviously incompatible with the above normalization, and these integrals are typically normalized by a condition like $\langle \psi_1 .. \psi_n \bar{\psi}_1 .. \bar{\psi}_m \rangle = 1$, where $n = \dim(ker(A))$ and $m = \dim(coker(A))$. Note also that the normalization condition C may not respect all symmetries of S, so that the final functional integral may in fact change by a scalar under such a symmetry transformation, which is exactly what is usually referred to under the name of an anomaly, so that this seems to be a good setting for their study.

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References

[1] F.A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Extensions of the Virasoro algebra constructed from Kac–Moody algebras using higher order Casimir invariants, Nucl. Phys. B 304 (1988) 348–370.
[2] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nucl. Phys. B 241 (1984) 333–380.
[3] M. Bershadsky, Superconformal algebras in two dimensions with arbitrary N, Phys. Lett. B 174 (1986) 285–288.
[4] S. Coleman, D. Gross and R. Jackiw, Fermion avatars of the Sugawara model, Phys. Rev. 180 (1969) 1359–1366.
[5] R.P. Feynman, The space–time approach to non-relativistic quantum mechanics, Rev. Modern Phys. 20 (1948) 367–387.
[6] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. 78 (1963) 267–288.
[7] A. Houriet and A. Kind, Classification invariante des termes de la matrice S. Helv. Phys. Acta 22 (1949) 319–330.
[8] K. Johnson and F.E. Low, Current algebra in a simple model, Progr. Theoret. Phys. Suppl. 37–38 (1966) 74–93.
[9] P. Jordan, Zur Herleitung der Vertauschungsregeln in der Neutrinotheorie des Lichtes, Z. Phys. 99 (1936) 109–113.

[10] V. Knizhnik, Superconformal algebras in two dimensions, Theoret. and Math. Phys. 66 (1986) 68–72.
[11] A.G. Knizhnik and A.B. Zamolodchikov, Current algebra and Wess–Zumino model in two dimensions, Nucl. Phys. B 247 (1984) 83–103.
[12] D. Kutasov, Geometry of the space of conformal field theories and contact terms, Phys. Lett. B 220 (1989) 153–158.
[13] J.H. Lowenstein, Normal products in the Thirring model, Comm. Math. Phys. 16 (1970) 265–289.
[14] R. Sasaki and I. Yamanaka, Virasoro algebra, Vertex operators, Quantum sine-Gordon and solvable Quantum Field Theories, Adv. Stud. Pure Math. 16 (1988) 271–296.
[15] H. Sugawara, A field theory of currents, Phys. Rev. 170 (1968) 1659–1662.
[16] G.C. Wick, The evaluation of the collision matrix, Phys. Rev. 80 (1950) 268–272.
[17] A.B. Zamolodchikov, Conformal symmetry and multicritical points in two-dimensional quantum field theory, Soviet. J. Nuclear Phys. 44 (1986) 529–533.